M3032006F1PV1

M303/J

Module Examination 2020

Further Pure Mathematics

Friday 12 June 2020 **10.00 am – 1.00 pm**

Time allowed: 3 hours

There are **two parts** in this examination.

In **Part 1** you should **attempt all 6 questions**. Each question is worth 11% of the total mark. You are advised to spend about two hours on this part.

In **Part 2** you should **attempt 2 out of the 3 questions**. Each question is worth 17% of the total mark.

Include all your working, as some marks are awarded for this.

Write your answers in the answer book(s) provided in **pen**, though you may draw diagrams in pencil.

Start your answer to each question on a new page, clearly indicating the number of the question.

Crossed out work will not be marked.

Write the numbers of the questions you have attempted in Part 2 on the front of your answer book(s).

Calculators are NOT permitted in this examination.

When sketching sets, take care to indicate which points are included: use dashed lines to indicate boundaries that are not included and solid lines to indicate boundaries that are. Use small open circles ∘ to indicate any 'corner' points on a boundary that are not included and small closed circles • to indicate any 'corner' points that are included.

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Part 1

You should **attempt all questions**.
Each question is worth 11%.

Question 1

(a) (i) Use the Euclidean Algorithm to find hcf(132, 77) (showing each step of your working). [3]

(ii) Hence find integers s and t such that $132s + 77t = \text{hcf}(132, 77)$ (showing each step of your working). [3]

(b) Determine the least positive integer that satisfies all three of the following linear congruences.

$$x \equiv 1 \pmod 3 \quad 2x \equiv 1 \pmod 7 \quad x + 1 \equiv 7 \pmod{10}$$ [4]

(c) Write down an expression for the congruence class that consists of all solutions of the system of linear congruences above. [1]

Question 2

(a) Let $G = \mathbb{Z}_4 \times \mathbb{Z}_{11}$.

(i) Write down the order of G. [1]

(ii) Write down the order of $(0, 3)$. [1]

(iii) Write down an element of order 22. [1]

(iv) Is G cyclic? Justify your answer. [2]

(b) Consider the dihedral group D_{22} given by

$$D_{22} = \langle r, s | r^{22} = s^2 = e, sr = r^{21}s \rangle.$$

(i) Write the element $r^{11}sr^{11}$ in the form $r^i s^j$ for some $i \in \{0, \ldots, 21\}$ and some $j \in \{0, 1\}$. [2]

(ii) Find the order of the element $r^{11}s$. [2]

(iii) Is the subgroup $\langle r^{11} \rangle$ normal in D_{22}? Justify your answer. [2]

Question 3

(a) Given $28 = 2^2 \times 7$, list all the divisors of 28.

Hence, or otherwise, find

(i) $\tau(28)$

(ii) $\sigma(28)$

(iii) $\phi(28)$ [3]

(b) Determine the following Legendre symbols (note that 2, 23 and 199 are prime).

(i) $(2/199)$

(ii) $(23/199)$ [3]

(c) Use the rational root test to show that

$$f(x) = x^3 - 4x^2 + 8x + 2$$

is irreducible over $\mathbb{Q}$. [5]

Question 4

Let

$$A = \{(a, b) \in \mathbb{R}^2 : 1 \leq a^2 + b^2 < 5\},\ B = \{(a, b) \in \mathbb{R}^2 : a < 0\}$$

and

$$C = A \cap B.$$

Let $d^{(2)}$ be the Euclidean metric on $\mathbb{R}^2$ given by:

$$d^{(2)}\big((x_1, x_2), (y_1, y_2)\big) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$.

(a) Sketch on separate diagrams or describe in words each of A, B and C taking care to indicate which points lie in the sets. [3]

(b) Determine whether $(0, 1)$ is a closure point of C for the Euclidean metric $d^{(2)}$. [2]

(c) Determine whether $(-1, 0)$ is an interior point of C for the Euclidean metric $d^{(2)}$. [2]

(d) Write down in set notation the closure, interior and boundary of C for the Euclidean metric $d^{(2)}$. [4]

Question 5

Let $\alpha = \sqrt{11} - \sqrt{3}$; you may assume that
$[\mathbb{Q}(\sqrt{11}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$, $\sqrt{11} \notin \mathbb{Q}(\sqrt{3})$ and $\sqrt{3} \notin \mathbb{Q}(\sqrt{11})$.

(a) Express α^3 in the form $a\sqrt{11} + b\sqrt{3}$ for suitable $a, b \in \mathbb{Q}$. [2]

(b) With the help of (a) or otherwise, show that $\sqrt{11}, \sqrt{3} \in \mathbb{Q}(\alpha)$. [3]

(c) Hence show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{11}, \sqrt{3})$. [1]

(d) Determine the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. [2]

(e) By considering α^2 find the minimal polynomial for α over $\mathbb{Q}$. [3]

Question 6

Define a metric, d, on $X = \mathbb{R}^2 - \{(a, b) : b = 0\}$ by

$$d(\mathbf{x}, \mathbf{y}) = 3|x_1 - y_1| + |x_2 - y_2|$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are points in $\mathbb{R}^2$.
(You do not have to show that d is a metric.)

(a) Write down $d\big((-1, 1), (1, 1)\big)$. [1]

(b) Write down a d-disconnection of X. [2]

(c) (i) Show that $(0, 1/n)$ is not a d-convergent sequence in X. [4]

(ii) Hence, or otherwise, show that X is not d-complete. [3]

(iii) Deduce that X is not d-sequentially compact. [1]

Part 2

You should **attempt two questions**. If you attempt more, the scores from your best two questions will count.
Each question is worth 17%.

Question 7

(a) Consider the following list of groups.

$$\mathbb{Z}_4 \times \mathbb{Z}_5; \quad D_5 \times \mathbb{Z}_2; \quad \mathbb{Z}_2 \times \mathbb{Z}_{10}; \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5; \quad \mathrm{Dic}_5$$

(i) Write down one non-abelian group from the list. [*1*]

(ii) Write down one cyclic group from the list. [*1*]

(iii) Write down one abelian group that is not cyclic from the list. [*1*]

(iv) Write down one pair of isomorphic groups from the list. [*1*]

(b) Let G be a group of order $153 = 17 \times 3^2$.

(i) Write down the order of the Sylow 3-subgroups. [*1*]

(ii) Show that G has a unique Sylow 17-subgroup and a unique Sylow 3-subgroup. [*3*]

(iii) Deduce that G is abelian. [*4*]

(iv) Write down the non-isomorphic possibilities for G. [*2*]

(c) Let G be any infinite group. Prove that G has a proper, non-trivial, subgroup. [*3*]

Question 8

Let

$$\mathbb{Z}[\rho] = \left\{a_0 + a_1\rho + a_2\rho^2 + \cdots + a_n\rho^n : a_i \in \mathbb{Z}, \text{ for } i = 0, 1, \ldots, n \text{ and } n \in \mathbb{N} \cup \{0\}\right\},$$

with $\rho = \frac{1}{2}\left(1 + \sqrt{-3}\right)$.

(a) Show that ρ satisfies $\rho^2 - \rho + 1 = 0$, and explain why this means that every element of $R = \mathbb{Z}[\rho]$ can be written in the form $a + b\rho$ for some $a, b \in \mathbb{Z}$. [*3*]

The set $\mathbb{Z}[\rho] = \{a + b\rho : a, b \in \mathbb{Z}\}$, where $\rho = \frac{1}{2}\left(1 + \sqrt{-3}\right)$ forms a ring with addition and multiplication defined in the natural way, namely:

$$(a_1 + b_1\rho) + (a_2 + b_2\rho) = (a_1 + a_2) + (b_1 + b_2)\rho$$

and

$$(a_1 + b_1\rho)(a_2 + b_2\rho) = a_1a_2 + (a_1b_2 + a_2b_1)\rho + b_1b_2\rho^2.$$

You may assume that the ring $\mathbb{Z}[\rho]$ is a Euclidean domain. You do not have to prove this.

Define a norm N on $\mathbb{Z}[\rho]$ by

$$N(a + b\rho) = a^2 + ab + b^2.$$

(You may assume that N is a multiplicative and Euclidean norm.)

(b) Show that an element $a + b\rho$ of $\mathbb{Z}[\rho]$ is a unit of $\mathbb{Z}[\rho]$ if and only if $(2a + b)^2 + 3b^2 = 4$.

Hence, or otherwise, prove that the only units in $\mathbb{Z}[\rho]$ are $\pm 1, \pm\rho, \pm(1 - \rho)$. [5]

(c) Find six elements $a + b\rho$ of $\mathbb{Z}[\rho]$ for which $N(a + b\rho) = 3$. [3]

(d) With the help of part (c) or otherwise, show that 3 is a reducible element. [2]

(e) Let $R = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$.

(i) Show that R is a subset of $\mathbb{Z}[\rho]$. [1]

(ii) For this part of the question you may assume that R is a subring of $\mathbb{Z}[\rho]$. Is 3 a reducible element in R? Briefly justify your answer. [3]

Question 9

(a) Let

$$A = \{f \in C[0,1] : |f(x) - f(y)| \leq (\sqrt{x} + \sqrt{y})|x - y| \text{ for each } x, y \in [0,1]\}$$

and

$$B = \{f \in C[0,1] : |f(x) - 2020| \leq 1 \text{ for } 0 \leq x \leq 1\}.$$

(i) Show that A is $d_{\max}$-closed. [3]

(ii) Deduce that $A \cap B$ is $d_{\max}$-closed. [1]

(iii) Hence or otherwise show that $A \cap B$ is $d_{\max}$-sequentially compact. You may assume that A is an equicontinuous set of functions. [2]

(b) Define a distance function $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as follows:

$$d(a,b) = \begin{cases} |\min\{a,b\}| + 2|\max\{a,b\}|, & \text{if } \min\{a,b\} \leq 0 < \max\{a,b\}, \\ |a - b|, & \text{if } \max\{a,b\} \leq 0, \\ 2|a - b|, & \text{if } \min\{a,b\} > 0. \end{cases}$$

(i) Write down $d(-2, 2)$. [1]

(ii) (1) Show that d satisfies (M1). [2]

(2) Show that d satisfies (M2). [2]

(3) Show that if $a, b, c \in \mathbb{R}$ with $a \leq 0$ and with $b, c > 0$, then $d(a, c) \leq d(a, b) + d(b, c)$. (In fact, by looking at all possible cases, one can verify that d satisfies (M3) (but we are not asking you to do this).) [3]

(c) Let (X, d) be a metric space and suppose that A and B are d-closed subsets of X. Prove directly from the definition of a d-closed set reproduced below that $A \cap B$ is also d-closed. You may not use any result in the Handbook after page 80. [*3*]

The definition of d-closed (from HB page 80) is as follows.

> Let (X, d) be a metric space and let $A \subseteq X$. We say that A is a d-closed set if for each d-convergent sequence with terms in A, the limit of the sequence is also in A. That is, whenever there is a d-convergent sequence (a_n) with $a_n \in A$ for each n then $\lim_{n\to\infty} a_n \in A$.

[END OF QUESTION PAPER]